Mathematical Methods in Physics HW8

1. Prove that $e^{z_1}e^{z_2} = e^{z_1+z_2}$.

Start with $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ where x_1, y_1, x_2, y_2 are real numbers.

$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2)$$

$$= e^{x_1+x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i[\cos y_1 \sin y_2 + \sin y_1 \cos y_2])$$

$$= e^{x_1+x_2}(\cos(y_1 + y_2) + i\sin[y_1 + y_2]) = e^{z_1+z_2}$$

2. Consider the following functions with $z = x + iy$, and determine where, if anywhere, they are differentiable and/or analytic.

a) $w(z) = \frac{x^6-y^4+i(x^5+x^3y^3+x^2y^2+y^5)}{x^3+y^2}$

First let's write:

$$w(z) = \frac{x^6-y^4}{x^3+y^2} + i\frac{x^5+x^3y^3+x^2y^2+y^5}{x^3+y^2} = \frac{(x^3-y^2)(x^3+y^2)}{x^3+y^2} + i\frac{(x^2+y^3)(x^3+y^2)}{x^3+y^2}$$

$$= x^3 - y^2 + i(x^2 + y^3) = u(x,y) + iv(x,y).$$

Then we should check the Cauchy-Riemann conditions: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

$$\left.\begin{array}{l}\frac{\partial u}{\partial x} = 3x^2 \\ \frac{\partial v}{\partial y} = 3y^2\end{array}\right\} \Rightarrow x^2 = y^2$$

$$\left.\begin{array}{l}\frac{\partial v}{\partial x} = 2x \\ \frac{\partial u}{\partial y} = 2y\end{array}\right\} \Rightarrow x = y$$

So these are differentiable along the line $x = y$. *Note: Both sets of conditions must be satisfied in order for differentiability.* However, if one takes a point on the line and considers the neighborhood, it must include points not on the line. Therefore this function is not analytic.

b) $w(z) = \frac{x+iy}{x+iy-1}$

To test for differentiability we need to check the Cauchy-Riemann equations.

First write:

$$w(z) = \frac{z}{z-1}\frac{(z^*-1)}{(z^*-1)} = \frac{zz^*-z}{zz^*-z^*-z+1} = \frac{(x+iy)(x-iy)-(x+iy)}{(x+iy)(x-iy)-(x-iy)-(x+iy)+1} = \frac{x^2+y^2-x-iy}{x^2+y^2-2x+1}$$

$$= \frac{x^2+y^2-x}{x^2+y^2-2x+1} - i\frac{y}{x^2+y^2-2x+1} = u(x,y) + iv(x,y)$$

Now we can check:

$\frac{\partial u}{\partial x} = \frac{2x-x^2+y^2-1}{(x^2+y^2-2x+1)^2} = \frac{\partial v}{\partial y}$ but neither exist at $x^2 + y^2 - 2x + 1 = 0 \Rightarrow y = 0, x = 1$ or $z = 1$.

So the function is not differentiable at this point.

Then of course $w(z) = \frac{z}{z-1}$ is the ratio of two functions of z (which are separately analytic) and hence is analytic as long as the denominator is not 0. So $w(z)$ is analytic everywhere except at $z = 1$ (where we would expect it to break down from the lack of differentiability).

c) $w(z) = \frac{|z|^2 - z^*}{z^*}$

First let's write:

$w(z) = \frac{zz^* - z^*}{z^*} = z - 1$ which is only a function of z with no singular points and is hence analytic everywhere and hence differentiable everywhere.

3. Consider $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$.

a) Determine the values of a, b, c, d such that the function is harmonic.

$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by$ and $\frac{\partial^2 u}{\partial y^2} = 2cx + 6dy$ hence

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = 6ax + 2by + 2cx + 6dy \Rightarrow 3a + c = 0,\ b + 3d = 0\ \Rightarrow a = -\frac{c}{3},\ b = -3d.$

Therefore: $u(x,y) = ax^3 - 3dx^2y - 3axy^2 + dy^3$

b) Find the harmonic conjugate of $u(x,y)$.

$\frac{\partial u}{\partial x} = 3ax^2 - 6dxy - 3ay^2 = \frac{\partial v}{\partial y} \quad \Rightarrow \quad v = 3ax^2y - 3dxy^2 - ay^3 + \varphi(x)$

$\frac{\partial u}{\partial y} = -3dx^2 - 6axy + 3dy^2$

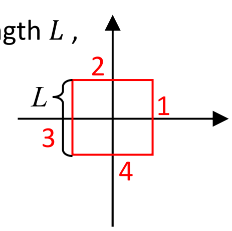
$-\frac{\partial v}{\partial x} = -6axy + 3dy^2 - \frac{\partial \varphi(x)}{\partial x} \quad \Rightarrow \quad \frac{\partial \varphi(x)}{\partial x} = 3dx^2 \quad \Rightarrow \varphi(x) = dx^3$

Thus: $v(x,y) = 3ax^2y - 3dxy^2 - ay^3 + dx^3$

c) Find an analytic function $w(z) = u(x,y) + iv(x,y)$ where $z = x + iy$.

$w(z) = ax^3 - 3dx^2y - 3axy^2 + dy^3 + i(3ax^2y - 3dxy^2 - ay^3 + dx^3) = az^3 + idz^3$

4. Consider integrating the function $w(z) = z^2$ around a contour that is a square of side length L, centered around the origin with sides parallel to the real and imaginary axes. That is:



a) Show explicitly that the result is independent of the length L.

First of all:

$w(z) = (x + iy)^2 = x^2 - y^2 + +i2xy \ \Rightarrow \ u(x,y) = x^2 - y^2, v(x,y) = 2xy$

And:

$\int_C w(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx)$

Breaking the integral up into four parts we have:

1. $x = \frac{L}{2}, dx = 0, y_i = -\frac{L}{2}, y_f = \frac{L}{2} \ \Rightarrow \ \int_{C_1} w(z)dz = -\int_{-\frac{L}{2}}^{\frac{L}{2}} Lydy + i\int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{L^2}{4} - y^2\right) dy$

2. $y = \frac{L}{2}, dy = 0, x_i = \frac{L}{2}, x_f = -\frac{L}{2} \ \Rightarrow \ \int_{C_2} w(z)dz = \int_{\frac{L}{2}}^{-\frac{L}{2}} \left(x^2 - \frac{L^2}{4}\right) dx + i\int_{\frac{L}{2}}^{-\frac{L}{2}} Lxdx$

3. $x = -\frac{L}{2}, dx = 0, y_i = \frac{L}{2}, y_f = -\frac{L}{2} \ \Rightarrow \ \int_{C_3} w(z)dz = \int_{\frac{L}{2}}^{-\frac{L}{2}} Lydy + i\int_{\frac{L}{2}}^{-\frac{L}{2}} \left(\frac{L^2}{4} - y^2\right) dy$

4. $y = -\frac{L}{2}, dy = 0, x_i = -\frac{L}{2}, x_f = \frac{L}{2} \Rightarrow \int_{C_3} w(z)dz = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(x^2 - \frac{L^2}{4}\right) dx - i \int_{-\frac{L}{2}}^{\frac{L}{2}} Lxdx$

Now note that all of the integrals of linear functions over the symmetric interval vanish. For the others they all come in cancelling pairs when we add up all contributions, e.g.

$$i \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{L^2}{4} - y^2\right) dy + i \int_{\frac{L}{2}}^{-\frac{L}{2}} \left(\frac{L^2}{4} - y^2\right) dy = i \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{L^2}{4} - y^2\right) dy - i \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{L^2}{4} - y^2\right) dy = 0$$

So in total we get 0, which is what we expect for the integral around a closed contour of an analytic function. So obviously it doesn't depend on L.

b) Use the Cauchy integral formula to find the value of $w(d)$ where d is real and $d < L$.

In principle we should evaluate $w(d) = \frac{1}{2\pi i} \oint_C \frac{w(z)}{z-d} dz = \frac{1}{2\pi i} \oint_C \frac{z^2}{z-d} dz$. However we can simplify things by relabeling $z \to z' = z - d \Rightarrow dz = dz'$ and hence our integral becomes

$w(d) = \frac{1}{2\pi i} \oint_C \frac{(z'+d)^2}{z'} dz' = \frac{1}{2\pi i} \left(\oint_C z'dz' + \oint_C d2dz' + \oint_C \frac{d^2}{z'} dz' \right)$

Note that the first two integrals are 0 since they are just the integral around a closed contour of analytic functions. For the third integral we may as well adopt polar coordinates and instead of the square contour, just take the circle of radius L. Then with $z' = Le^{i\theta} \Rightarrow dz' = iLe^{i\theta} d\theta$ and we have:

$w(d) = \frac{d^2}{2\pi} \int_0^{2\pi} d\theta = d^2$ as expected!

c) Does the Cauchy integral formula work for $d > L$? Why or why not?

Yes it does! This is obvious from the fact that L dropped completely out of our work in part (b). But also because you can think of z^2 as an entire function which means that we can interpret the contour as surrounding 0 or ∞. Then the Cauchy integral theorem applies for objects being "outside" of the contour surrounding 0 because they are "inside" the contour around ∞.